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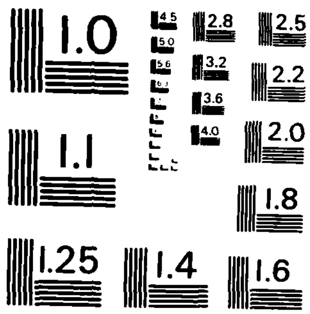

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ANALYTICAL EVALUATION OF THE POTENTIAL DUE TO A UNIFORMLY CHARGED PLANAR TRIANGULAR REGION

by

Joseph R. Mautz
Ecument Arvas
Roger F. Harrington

Department of
Electrical and Computer Engineering
Syracuse University
Syracuse, New York 13210

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The electrostatic potential due to a planar triangular patch of surface charge of unit charge density in free space is proportional to the surface integral over the patch of the reciprocal of the distance to the observation point. Several closed form expressions for this integral are available in the literature. The present report gives a detailed derivation of one of these expressions.

CONTENTS

I. INTRODUCTION

The electrostatic potential due to a planar triangular patch of surface charge of unit charge density in free space is called V and is given by

$$V(\underline{r}) = \frac{1}{4\pi\varepsilon_o} \iint_T \frac{ds'}{|\underline{r} - \underline{r}'|} \tag{1}$$

Here, T denotes the surface of the triangular patch, ds' is a differential element of surface area on T, $\underline{r}'$ is the radius vector to the location of ds', $\underline{r}$ is the radius vector to the point at which V is evaluated, and ε_o is the permittivity of free space. The potential integral (1) arises in numerical solutions of electrostatic, magnetostatic, and electromagnetic problems involving arbitrarily shaped surfaces modeled by planar triangular subdomains [1], [2].

The potential integral (1) has been evaluated by several authors [1], [3]-[6]. It is perplexing to note that the form of V obtained in any one of the five references [1] and [3]-[6] is different from that obtained in any of the others. Although the form obtained in [6] is perhaps the most convenient for numerical work, the objective of the present report is to give a detailed derivation of the form obtained in [5]. This form is [5, Eq. (4)].

II. TRANSFORMATION TO AREA COORDINATES

In this section, the integral on the right-hand side of (1) is expressed as an iterated integral with respect to two area coordinates.

As shown in Fig. 1, the triangle T has vertices $\underline{r}_1$, $\underline{r}_2$, and $\underline{r}_3$. In Fig. 1, the point $\underline{r}'$ on T is called P', and T is divided into the three subareas A_1, A_2, and A_3. To facilitate the evaluation of $V(\underline{r})$, area coordinates (ξ,η,ζ) are defined by

$$\xi = \frac{A_1}{A}, \qquad \eta = \frac{A_2}{A}, \qquad \zeta = \frac{A_3}{A} \tag{2}$$

where A is the total area of the triangle T.

It is now shown that

$$\underline{r}' = (1-\eta-\zeta)\underline{r}_1 + \eta\underline{r}_2 + \zeta\underline{r}_3 \tag{3}$$

In order to obtain (3), we express $\underline{r}'$ as

$$\underline{r}' = \underline{r}_1 + \underline{R}' \tag{4}$$

where, as shown in Fig. 2, $\underline{R}'$ is the vector drawn from vertex 1 to the point P'. The dotted line in Fig. 2 passes through the point P', is parallel to the line 13, and intersects the line 12 at the point C. It is obvious from Fig. 2 that

$$\underline{R}' = \overline{1C} + \overline{CP'} \tag{5}$$

where $\overline{1C}$ is the vector drawn from vertex 1 to the point C and $\overline{CP'}$ is the vector drawn from C to P'. Equation (5) is recast as

$$\underline{R}' = \frac{1C}{|\underline{r}_2 - \underline{r}_1|}(\underline{r}_2-\underline{r}_1) + \frac{CP'}{|\underline{r}_3 - \underline{r}_1|}(\underline{r}_3-\underline{r}_1) \tag{6}$$

where 1C is the length of the vector $\overline{1C}$ and CP' is the length of the vector $\overline{CP'}$. From Fig. 2, it is easy to see that

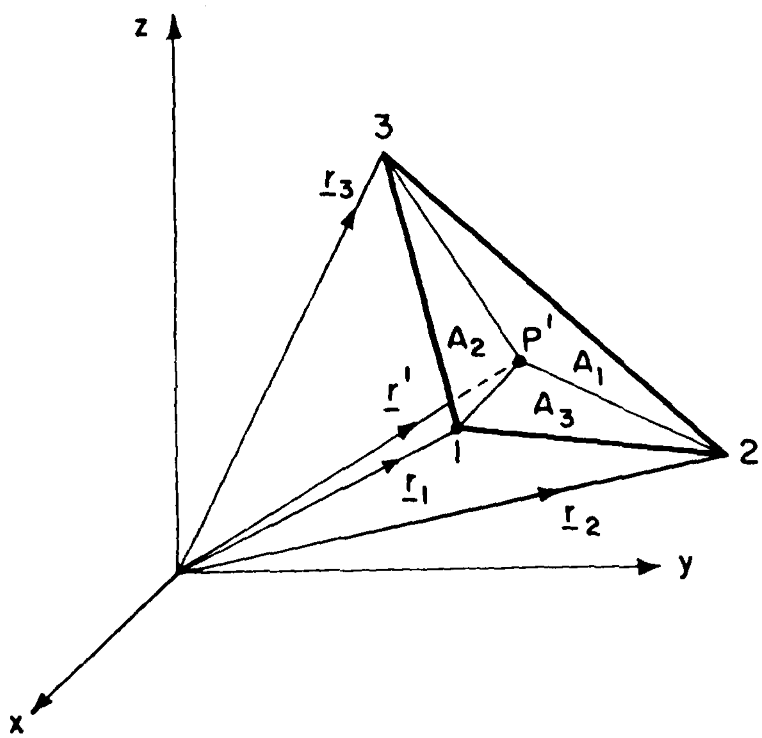

Fig. 1. The triangle T with vertices $\underline{r}_1$, $\underline{r}_2$, and $\underline{r}_3$.

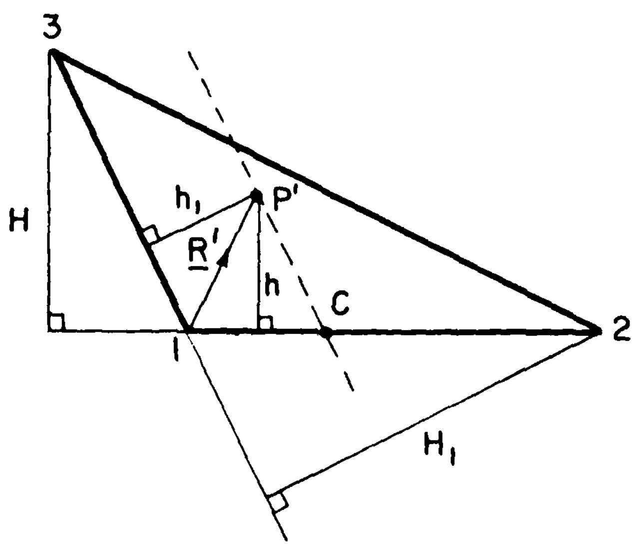

Fig. 2. Geometry for the evaluation of $\underline{R}'$.

$$\frac{1C}{|\underline{r}_2 - \underline{r}_1|} = \frac{h_1}{H_1} = \frac{h_1|\underline{r}_3 - \underline{r}_1|}{H_1|\underline{r}_3 - \underline{r}_1|} = \frac{A_2}{A} = \eta \tag{7}$$

and

$$\frac{CP'}{|\underline{r}_3 - \underline{r}_1|} = \frac{h}{H} = \frac{h|\underline{r}_2 - \underline{r}_1|}{H|\underline{r}_2 - \underline{r}_1|} = \frac{A_3}{A} = \zeta \tag{8}$$

Hence, (6) becomes

$$\underline{R}' = \eta(\underline{r}_2-\underline{r}_1) + \zeta(\underline{r}_3-\underline{r}_1) \tag{9}$$

Substitution of (9) into (4) gives the desired result (3).

In regard to the differential element of area ds' in (1), Fig. 3 shows a finite element of area Δs. The area Δs is the area of the parallelogram bounded by the lines corresponding to η, $\eta+\Delta\eta$, ζ, and $\zeta+\Delta\zeta$. The vertices of this parallelogram are the points $a(\eta, \zeta+\Delta\zeta)$, $b(\eta+\Delta\eta,\zeta+\Delta\zeta)$, $c(\eta,\zeta)$, and $d(\eta+\Delta\eta,\zeta)$. The area of this parallelogram is given by

$$\Delta s = |\overline{ac} \times \overline{ab}| \tag{10}$$

where $\overline{ac}$ and $\overline{ab}$ are the vectors drawn from the point a to the points c and b, respectively. Specifically,

$$\overline{ac} = \underline{r}'(\eta,\zeta) - \underline{r}'(\eta,\zeta+\Delta\zeta) \tag{11}$$

Substitution of (3) into (11) gives

$$\overline{ac} = \Delta\zeta(\underline{r}_1-\underline{r}_3) \tag{12}$$

Similarly,

$$\overline{ab} = \Delta\eta(\underline{r}_2-\underline{r}_1) \tag{13}$$

Substitution of (12) and (13) into (10) gives

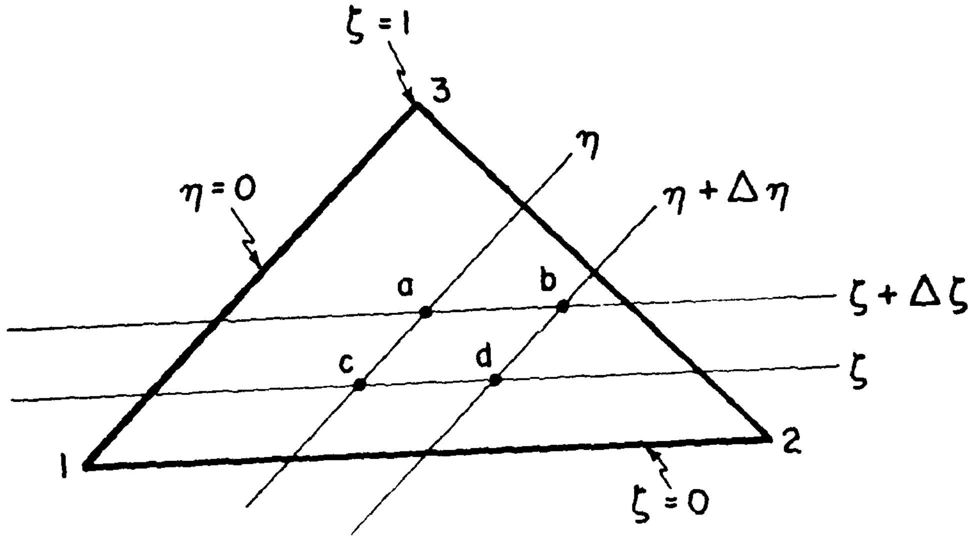

Fig. 3. Representation of Δs as the area of the parallelogram whose vertices are a, b, c, and d.

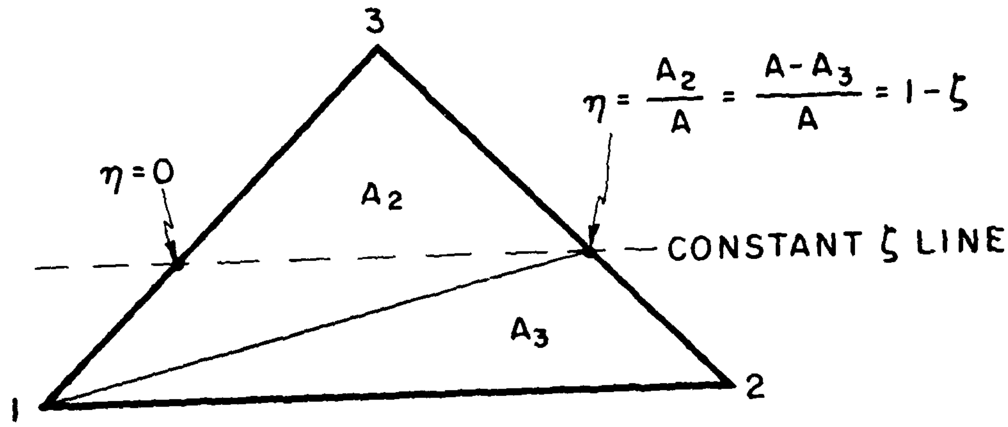

Fig. 4. Range of η along a line of constant ζ .

$$\Delta s = |(\underline{r}_2-\underline{r}_1) \times (\underline{r}_3-\underline{r}_1)|\Delta\zeta\Delta\eta \tag{14}$$

Thanks to (14), (1) becomes

$$V(\underline{r}) = \frac{|(\underline{r}_2-\underline{r}_1) \times (\underline{r}_3-\underline{r}_1)|}{4\pi\varepsilon_o} \int_0^1 d\zeta \int_0^{1-\zeta} \frac{d\eta}{R} \tag{15}$$

where

$$R = |\underline{r} - \underline{r}'| \tag{16}$$

and $\underline{r}'$ is given by (3). The limits of integration in (15) were obtained from Fig. 4.

III. THE INTEGRAL WITH RESPECT TO η

The integral with respect to η in (15) is evaluated in this section.

As the first step in the evaluation of this integral, R is expressed in terms of η and ζ. Substitution of (3) for $\underline{r}'$ in expression (16) for R and subsequent squaring give

$$R^2 = |\eta(\underline{r}_2-\underline{r}_1) + \zeta(\underline{r}_3-\underline{r}_1) - (\underline{r}-\underline{r}_1)|^2 \tag{17}$$

which becomes

$$R^2 = \alpha\eta^2 + 2\beta\eta + \gamma \tag{18}$$

where

$$\left.\begin{aligned} \alpha &= |\underline{r}_2 - \underline{r}_1|^2 \\ \beta &= (\underline{r}_2-\underline{r}_1) \cdot (\zeta(\underline{r}_3-\underline{r}_1) - (\underline{r}-\underline{r}_1)) \\ \gamma &= |\zeta(\underline{r}_3-\underline{r}_1) - (\underline{r}-\underline{r}_1)|^2 \end{aligned}\right\} \tag{19}$$

Completing the square in (18), we obtain

$$R^2 = \alpha((\eta + \frac{\beta}{\alpha})^2 + \frac{\alpha\gamma - \beta^2}{\alpha^2}) \tag{20}$$

Being the square of the length of one of the sides of the triangle T, $\alpha > 0$. Now, thanks to (20),

$$\int_0^{1-\zeta} \frac{d\eta}{R} = \frac{1}{\sqrt{\alpha}} \int_0^{1-\zeta} \frac{d\eta}{\sqrt{(\eta + \frac{\beta}{\alpha})^2 + \frac{\alpha\gamma - \beta^2}{\alpha^2}}} \tag{21}$$

Since, from (17), $R^2 \geq 0$, it is evident from (20) that

$$(\eta + \frac{\beta}{\alpha})^2 + \frac{\alpha\gamma - \beta^2}{\alpha^2} \geq 0 \tag{22}$$

for all finite values of η and ζ. In particular, (22) is true for $\eta = -\beta/\alpha$ so that

$$\alpha\gamma - \beta^2 \geq 0 \tag{23}$$

It is evident from (19) that $\alpha\gamma - \beta^2$ is a quadratic function of ζ where the coefficient of ζ^2 is

$$|\underline{r}_2 - \underline{r}_1|^2 \; |\underline{r}_3 - \underline{r}_1|^2 - ((\underline{r}_2 - \underline{r}_1) \cdot (\underline{r}_3 - \underline{r}_1))^2$$

The above quantity is positive because the vectors $(\underline{r}_2 - \underline{r}_1)$ and $(\underline{r}_3 - \underline{r}_1)$, being two sides of the triangle T, can not be parallel to each other. So far, it has been established $\alpha\gamma - \beta^2$ satisfies the inequality (23) and is a quadratic function of ζ where the coefficient of ζ^2 is positive. Consequently, there can not be more than one value of ζ for which

$$\alpha\gamma - \beta^2 = 0 \tag{24}$$

Hence, as far as the integration with respect to ξ in (15) is concerned, we may, when integrating with respect to η, assume that

$$\alpha\gamma - \beta^2 > 0 \tag{25}$$

In view of (25), application of [7, Formula 200.01.] to the integral on the right-hand side of (21) gives

$$\int_0^{1-\xi} \frac{d\eta}{R} = \frac{1}{\sqrt{\alpha}} \left[\ln\left(\eta + \frac{\beta}{\alpha} + \sqrt{\left(\eta + \frac{\beta}{\alpha}\right)^2 + \frac{\alpha\gamma - \beta^2}{\alpha^2}}\right)\right]_0^{1-\xi} \tag{26}$$

Multiplication of the argument of the logarithm in (26) by any positive constant will not change the value of (26). Choosing to multiply by $\sqrt{\alpha}$ and evaluating the right-hand side of (26) at the limits 0 and $1-\xi$, we obtain

$$\int_0^{1-\xi} \frac{d\eta}{R} = \frac{1}{\sqrt{\alpha}} \left[\ln\left(R_2 + x_2\right) - \ln\left(R_1 + x_1\right)\right] \tag{27}$$

where

$$R_1 = \sqrt{\left(\frac{\beta}{\sqrt{\alpha}}\right)^2 + \frac{\alpha\gamma - \beta^2}{\alpha}} \tag{28}$$

$$R_2 = \sqrt{\left(\frac{\alpha(1-\xi) + \beta}{\sqrt{\alpha}}\right)^2 + \frac{\alpha\gamma - \beta^2}{\alpha}} \tag{29}$$

$$x_1 = \frac{\beta}{\sqrt{\alpha}} \tag{30}$$

$$x_2 = \frac{\alpha(1-\xi) + \beta}{\sqrt{\alpha}} \tag{31}$$

According to (20), the argument of the square root in (28) is R^2 at $\eta = 0$, and the argument of the square root in (29) is R^2 at $\eta = 1-\zeta$. Now, use of (17) for R^2 reduces (28) and (29) to

$$R_i = \sqrt{A_i \zeta^2 + B_i \zeta + C_i} \quad , \qquad i = 1,2 \tag{32}$$

where

$$A_i = |\underline{r}_3 - \underline{r}_i|^2 \tag{33}$$

$$B_i = -2(\underline{r}_3 - \underline{r}_i) \cdot (\underline{r} - \underline{r}_i) \tag{34}$$

$$C_i = |\underline{r} - \underline{r}_i|^2 \tag{35}$$

Substitution of (19) for ξ and ζ in (30) and (31) gives

$$x_i = D_i \zeta + E_i \quad , \qquad i = 1,2 \tag{36}$$

where

$$D_i = \frac{(\underline{r}_2 - \underline{r}_1) \cdot (\underline{r}_3 - \underline{r}_i)}{|\underline{r}_2 - \underline{r}_1|} \tag{37}$$

$$E_i = -\frac{(\underline{r}_2 - \underline{r}_1) \cdot (\underline{r} - \underline{r}_i)}{|\underline{r}_2 - \underline{r}_1|} \tag{38}$$

IV. INEQUALITIES FOR A_i, B_i, C_i, D_i, AND E_i

Because $\underline{r}_1$, $\underline{r}_2$, and $\underline{r}_3$ are the vertices of the triangle T, A_i of (33) satisfies the inequality

$$A_i > 0, \quad i = 1,2 \tag{39}$$

Being two sides of the triangle T, the vectors $\underline{r}_2 - \underline{r}_1$ and $\underline{r}_3 - \underline{r}_i$ can never be parallel to each other. Hence, it is evident from (33) and (37) that

$$A_i - D_i^2 > 0 \ , \quad i = 1,2 \tag{40}$$

For later convenience, we desire to have

$$D_i \neq 0 \ , \quad i = 1,2 \tag{41}$$

Since the triangle T can not have two right angles, it is evident from (37) that $D_1 = D_2 = 0$ is impossible. If either D_1 or D_2 is zero, then we can cyclically permute the assignment of the vectors $\underline{r}_1$, $\underline{r}_2$, and $\underline{r}_3$ to the vertices of the triangle T until neither D_1 nor D_2 is zero. Hence we can, without loss of generality, assume that (41) holds.

It is evident from (23), (28), (29), and (32) that

$$R_i^2 = A_i\zeta^2 + B_i\zeta + C_i \geq 0 \qquad -\infty < \zeta < \infty \tag{42}$$

Now, R_i^2 can be expressed as

$$R_i^2 = A_i\left[\left(\zeta + \frac{B_i}{2A_i}\right)^2 + \frac{4A_iC_i - B_i^2}{4A_i^2}\right] \tag{43}$$

Clearly, the minimum value of R_i^2 with respect to ζ in (43) is given by

$$\min(R_i^2) = \frac{4A_iC_i - B_i^2}{4A_i} \tag{44}$$

where min denotes minimum value. In view of (39), it is apparent from (42) and (44) that

$$4A_iC_i - B_i^2 \geq 0 \tag{45}$$

The definitions (33) - (35) of A_i, B_i, and C_i reveal that equality is attained in (45) only when the point $\underline{r}$ lies on the line which passes through the points $\underline{r}_3$ and $\underline{r}_i$.

Inspection of (28)-(31) reveals that

$$R_i^2 - x_i^2 = \frac{\alpha\gamma - \beta^2}{\alpha} \tag{46}$$

Taking the minimum with respect to ζ of both sides of (46), we obtain

$$\min(R_i^2 - x_i^2) = \min\left(\frac{\alpha\gamma - \beta^2}{\alpha}\right) \tag{47}$$

Since α is positive, it is apparent from (20) that the minimum of R^2 with respect to η and ζ is the minimum of $(\alpha\gamma - \beta^2)/\alpha$ with respect to ζ.

$$\min(R^2) = \min\left(\frac{\alpha\gamma - \beta^2}{\alpha}\right) \tag{48}$$

As a consequence of (47) and (48),

$$\min(R_i^2 - x_i^2) = \min(R^2) \tag{49}$$

If the point $\underline{r}$ is not in the plane of the triangle T, then $\underline{r} - \underline{r}_1$ can not be expressed as a linear combination of the vectors $\underline{r}_2 - \underline{r}_1$ and $\underline{r}_3 - \underline{r}_1$ which appear in (17). As a result, the vector subject to the magnitude operation in (17) can never be zero so that

$$\min(R^2) > 0 \tag{50}$$

As a consequence of (49) and (50),

$$\min(R_i^2 - x_i^2) > 0 \tag{51}$$

The definitions (32) and (36) of R_i and x_i lead to

$$R_i^2 - x_i^2 = (A_i - D_i^2)\left(\zeta^2 + \frac{B_i - 2D_iE_i}{A_i - D_i^2}\,\zeta + \frac{C_i - E_i^2}{A_i - D_i^2}\right) \tag{52}$$

Upon completion of the square, (52) becomes

$$R_i^2 - x_i^2 = (A_i - D_i^2)\left[\left(\xi + \frac{B_i - 2D_iE_i}{2(A_i - D_i^2)}\right)^2 + \frac{d_i^2}{D_i^2}\right] \tag{53}$$

where

$$d_i^2 = D_i^2\left[\frac{C_i - E_i^2}{A_i - D_i^2} - \frac{(B_i - 2D_iE_i)^2}{4(A_i - D_i^2)^2}\right] \tag{54}$$

In view of (40) and (41), it is evident from (51) and (53) that

$$d_i^2 > 0 \tag{55}$$

The above inequality holds whenever the point $\underline{r}$ is not in the plane of the triangle T.

If the point $\underline{r}$ is in the plane of the triangle T, then $\underline{r} - \underline{r}_1$ can be expressed as a linear combination of the vectors $\underline{r}_2 - \underline{r}_1$ and $\underline{r}_3 - \underline{r}_1$ which appear in (17). Consequently,

$$\min(R^2) = 0 \tag{56}$$

Retracing the development (50)-(55) with (50) replaced by (56), we obtain

$$d_i^2 = 0 \tag{57}$$

Equation (57) holds whenever the point $\underline{r}$ is in the plane of the triangle T.

In view of (19), substitution of (27) into (15) gives

$$V(\underline{r}) = \frac{|(\underline{r}_2 - \underline{r}_1) \times (\underline{r}_3 - \underline{r}_1)|}{4\pi\epsilon_0 |\underline{r}_2 - \underline{r}_1|} \sum_{i=1}^{2} (-1)^i \int_0^1 [\ln(R_i + x_i)]d\xi \tag{58}$$

The integral in (58) will be evaluated next.

Instead of the precise integral in (58), we consider, for generality and notational simplicity, the indefinite integral I_1 defind by

$$I_1 = \int [\ln(V + v)]dx \tag{59}$$

where

$$V = \sqrt{Ax^2 + Bx + C} \tag{60}$$

$$v = Dx + E \tag{61}$$

Here, A,B,C,D, and E represent the previously defined constants A_i, B_i, C_i, D_i, and E_i, respectively. From (39), (40), (41), (45), (54), (55), and (57), we have

$$A > 0 \tag{62}$$

$$A - D^2 \neq 0 \tag{63}$$

$$D \neq 0 \tag{64}$$

$$4AC - B^2 \geq 0 \tag{65}$$

$$d^2 \geq 0 \tag{66}$$

where

$$d^2 = D^2\left[\frac{C - E^2}{A - D^2} - \frac{(B - 2DE)^2}{4(A - D^2)^2}\right] \tag{67}$$

V. THE INTEGRAL OF THE LOGARITHM; $4AC - B^2 = 0$

In this section, the integral I_1 of (59) is evaluated for the case in which

$$4AC - B^2 = 0 \tag{68}$$

Equation (67) can be rewritten as

$$d^2 = \frac{D^2}{4A(A - D^2)^2} [(A - D^2)(4AC - B^2) - (2AE - BD)^2] \tag{69}$$

In view of (66) and (69), it is apparent that (68) is accompanied by

$$2AE - BD = 0 \tag{70}$$

$$d^2 = 0 \tag{71}$$

Completing the square on the right-hand side of (60), we obtain

$$V = \sqrt{A} \sqrt{(x + \frac{B}{2A})^2 + \frac{4AC - B^2}{4A^2}} \tag{72}$$

Equation (68) reduces (72) to

$$V = \sqrt{A} \left| x + \frac{B}{2A} \right| \tag{73}$$

Substitution of (73) and (61) into (59) gives

$$I_1 = \int [\ln(\sqrt{A} \left| x + \frac{B}{2A} \right| + Dx + E)]dx \tag{74}$$

Assuming that $x + B/(2A)$ does not change sign during the integration in (74), we obtain [7, Formula 610.]

$$I_1 = - \frac{(V + v)\ln(V + v) - (V + v)}{\sqrt{A} - D}, \quad x + \frac{B}{2A} < 0 \tag{75}$$

$$I_1 = \frac{(V + v)\ln(V + v) - (V + v)}{\sqrt{A} + D}, \quad x + \frac{B}{2A} > 0 \tag{76}$$

The combination of (75) and (76) will be a valid representation for the integral (59) only if this combination is continuous at

$$x + \frac{B}{2A} = 0$$

According to (61) and (73)

$$V + v = \sqrt{A}\,\left|x + \frac{B}{2A}\right| + D\left(x + \frac{B}{2A}\right) + \frac{2AE - BD}{2A} \tag{77}$$

Equation (70) reduces (77) to

$$V + v = \sqrt{A}\,\left|x + \frac{B}{2A}\right| + D\left(x + \frac{B}{2A}\right) \tag{78}$$

It is evident from (78) that $V + v$ approaches zero as

$$x + \frac{B}{2A} \to 0$$

Therefore, the right-hand sides of both (75) and (76) approach zero as

$$x + \frac{B}{2A} \to 0$$

Consequently, the combination of (75) and (76) is continuous at

$$x + \frac{B}{2A} = 0$$

Hence, the combination of (75) and (76) is a valid representation for the integral (59). This combination is expressed as

$$I_1 = \frac{2Ax + B}{2A} \ln(V + v) - x - \frac{B}{2A} \tag{79}$$

Omitting the constant $-B/(2A)$ from (79), we obtain

$$I_1 = \frac{2Ax + B}{2A} \ln(V + v) - x \tag{80}$$

Expression (80) is I_1 of (59) for the case in which (68) holds.

VI. THE INTEGRAL OF THE LOGARITHM; $4AC - B^2 > 0$, $d^2 = 0$

In this section, the integral I_1 of (59) is evaluated for the case in which

$$4AC - B^2 > 0 \tag{81}$$

$$d^2 = 0 \tag{82}$$

The substitution [8, Sec. 2.251]

$$V = t - \sqrt{A}\, x \tag{83}$$

is used.

Recalling that V is given by (60), we square both sides of (83) to obtain

$$Bx + C = t^2 - 2\sqrt{A}\, xt \tag{84}$$

and we find that

$$x = \frac{t^2 - C}{2\sqrt{A}\, t + B} \tag{85}$$

$$dx = \frac{2(\sqrt{A}t^2 + Bt + \sqrt{A}\, C)}{(2\sqrt{A}t + B)^2}\, dt \tag{86}$$

$$V = \frac{\sqrt{A}\, t^2 + Bt + \sqrt{A}\, C}{2\sqrt{A}\, t + B} \tag{87}$$

$$V + v = \frac{(\sqrt{A} + D)t^2 + (2\sqrt{A}\, E + B)t + BE + (\sqrt{A} - D)C}{2\sqrt{A}t + B} \tag{88}$$

Upon completion of the square in the numerator, (88) becomes

$$V + v = \frac{(\sqrt{A} + D)\left[\left(t + \frac{2\sqrt{A}\, E + B}{2(\sqrt{A} + D)}\right)^2 + \frac{d^2(\sqrt{A} - D)^2}{D^2}\right]}{2\sqrt{A}\, t + B} \tag{89}$$

where

$$d^2 = \frac{D^2}{(\sqrt{A} - D)^2} \left[\frac{BE + (\sqrt{A} - D)C}{\sqrt{A} + D} - \frac{(2\sqrt{A}E + B)^2}{4(\sqrt{A} + D)^2}\right] \tag{90}$$

It is easy to show that the right-hand side of (90) is equal to that of (69).

Expressions (85) - (89) would not be well-behaved if the quantity $2\sqrt{A}\, t + B$ could pass through zero. This quantity will never be zero if it can be shown that

$$t + \frac{B}{2\sqrt{A}} > 0 \tag{91}$$

The inequality (91) is established in the following manner. From (83),

$$t = V + \sqrt{A}\, x \tag{92}$$

Substitution of (72) into (92) and subsequent addition of $B/(2\sqrt{A})$ to both sides of (92) give

$$t + \frac{B}{2\sqrt{A}} = \sqrt{A}\left(\sqrt{\left(x + \frac{B}{2A}\right)^2 + \frac{4AC - B^2}{4A^2}} + x + \frac{B}{2A}\right) \tag{93}$$

In view of (62) and (81), the right-hand side of (93) is positive. Hence, the inequality (91) holds. Consequently, expressions (85)-(89) are well-behaved.

Thanks to (82), substitution of (86) and (89) into (59) gives

$$I_1 = 2 \int \frac{\sqrt{A}\, t^2 + Bt + \sqrt{A}\, C}{(2\sqrt{A}\, t + B)^2} \ln w \, dt \tag{94}$$

where

$$w = \frac{\sqrt{A} + D}{2\sqrt{A}\, t + B} \left(t + \frac{2\sqrt{A}\, E + B}{2(\sqrt{A} + D)}\right)^2 \tag{95}$$

The identity

$$\frac{\sqrt{A}\,t^2 + Bt + \sqrt{A}\,C}{(2\sqrt{A}\,t + B)^2} = \frac{1}{4\sqrt{A}}\left(1 + \frac{4AC - B^2}{4A(t + \frac{B}{2\sqrt{A}})^2}\right) \tag{96}$$

reduces (94) to

$$I_1 = \frac{1}{2\sqrt{A}}\left(I_2 + \frac{4AC - B^2}{4A} I_3\right) \tag{97}$$

where

$$I_2 = \int \ln w \, dt \tag{98}$$

$$I_3 = \int \frac{\ln w}{(t + \frac{B}{2\sqrt{A}})^2} \, dt \tag{99}$$

Use of (95) leads to

$$\ln w = \ln\left(\frac{\sqrt{A} + D}{2\sqrt{A}}\right) - \ln\left(t + \frac{B}{2\sqrt{A}}\right) + 2 \ln\left|t + \frac{2\sqrt{A}\,E + B}{2(\sqrt{A} + D)}\right| \tag{100}$$

Thanks to the integration formula [7, Formula 610.]

$$\int \ln|x|\,dx = x \ln|x| - x$$

the integral (98) becomes

$$I_2 = t \ln\left(\frac{\sqrt{A} + D}{2\sqrt{A}}\right) - w_1 \ln w_1 + w_1 + 2w_2 \ln|w_2| - 2w_2 \tag{101}$$

where

$$w_1 = t + \frac{B}{2\sqrt{A}} \tag{102}$$

$$w_2 = t + \frac{2\sqrt{A}\,E + B}{2(\sqrt{A} + D)} \tag{103}$$

Upon substitution of (92) for t, (101) becomes

$$I_2 = (V + \sqrt{A}\,x)\ln\left(\frac{\sqrt{A} + D}{2\sqrt{A}}\right) - w_1 \ln w_1 + w_1 + 2w_2 \ln|w_2| - 2w_2 \tag{104}$$

where

$$w_1 = V + \frac{2Ax + B}{2\sqrt{A}} \tag{105}$$

$$w_2 = V + \frac{2Ax + B}{2\sqrt{A}} + \frac{2AE - BD}{2\sqrt{A}(\sqrt{A} + D)} \tag{106}$$

Substituting (100) into (99) and changing the variable of integration from t to w_1 of (102), we obtain

$$I_3 = -\frac{1}{w_1}\ln\left(\frac{\sqrt{A} + D}{2\sqrt{A}}\right) - \int \frac{\ln w_1}{w_1^2}\,dw_1 + 2\int \frac{1}{w_1^2}\ln\left|w_1 + \frac{2AE - BD}{2\sqrt{A}(\sqrt{A} + D)}\right| dw_1 \tag{107}$$

The integrals in (107) are evaluated using the integration formulas [7, Formulas 611.2. and 621.2.]

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} \tag{108}$$

$$\int \frac{\ln|a+bx|}{x^2}\,dx = \frac{b}{a}\ln x - \left(\frac{1}{x} + \frac{b}{a}\right)\ln|a+bx| \tag{109}$$

The result is

$$I_3 = -\frac{1}{w_1}\ln\left(\frac{\sqrt{A} + D}{2\sqrt{A}}\right) + \frac{\ln w_1}{w_1} + \frac{1}{w_1} + \frac{4\sqrt{A}\,(\sqrt{A} + D)}{2AE - BD}\ln w_1$$

$$-2\left(\frac{1}{w_1} + \frac{2\sqrt{A}\,(\sqrt{A} + D)}{2AE - BD}\right)\ln|w_2| \tag{110}$$

where w_1 and w_2 are given by (105) and (106). Equations (69), (81), and (82) assure that the denominator $2AE - BD$ in (110) is not zero.

Substitution of (104) and (110) into (97) gives

$$I_1 = \frac{1}{2\sqrt{A}}[(w_1 - 2w_2 + \frac{4AC - B^2}{4A\ w_1}) + (V + \sqrt{A}\ x - \frac{4AC - B^2}{4A\ w_1})\ \ln\ (\frac{\sqrt{A} + D}{2\sqrt{A}})$$

$$+(-w_1 + \frac{4AC - B^2}{4A}\ (\frac{1}{w_1} + \frac{4\sqrt{A}\ (\sqrt{A} + D)}{2AE - BD}))\ln\ w_1$$

$$+ 2(w_2 - \frac{4AC - B^2}{4A}\ (\frac{1}{w_1} + \frac{2\sqrt{A}\ (\sqrt{A} + D)}{2AE - BD}))\ \ln|w_2|] \quad (111)$$

With V, w_1, and w_2 given by (60), (105), and (106), simple algebra reduces (111) to

$$I_1 = \frac{1}{2\sqrt{A}}\ [(-\ \frac{2Ax + B}{\sqrt{A}} - \frac{2AE - BD}{\sqrt{A}\ (\sqrt{A} + D)}) + \frac{4Ax + B}{2\sqrt{A}}\ \ln\ (\frac{\sqrt{A} + D}{2\sqrt{A}})$$

$$-\ (\frac{2Ax + B}{\sqrt{A}} - \frac{(4AC - B^2)(\sqrt{A} + D)}{\sqrt{A}\ (2AE - BD)})\ \ln\ w_1$$

$$+ 2(\frac{2Ax + B}{\sqrt{A}} + \frac{2AE - BD}{2\sqrt{A}\ (\sqrt{A} + D)} - \frac{(4AC - B^2)\ (\sqrt{A} + D)}{2\sqrt{A}\ (2AE - BD)})\ \ln|w_2|] \quad (112)$$

From (69) and (82), we obtain

$$4AC - B^2 = \frac{(2AE - BD)^2}{A - D^2} \quad (113)$$

Substitution of (113) for $4AC - B^2$ in (112) gives

$$I_1 = -x - \frac{2\sqrt{A}\ E + B}{2\sqrt{A}\ (\sqrt{A} + D)} + (x + \frac{B}{4A})\ \ln\ (\frac{\sqrt{A} + D}{2\sqrt{A}})$$

$$-\ (x - \frac{2\sqrt{A}\ E - B}{2\sqrt{A}\ (\sqrt{A} - D)})\ \ln\ w_1 + 2(x + \frac{B - 2DE}{2(A - D^2)})\ln|w_2| \quad (114)$$

Expression (114) is recast as

$$I_1 = \left(x + \frac{B - 2DE}{2(A - D^2)}\right)\left(\ln\left(\frac{\sqrt{A} + D}{2\sqrt{A}}\right) - \ln w_1 + 2\ln|w_2|\right)$$

$$+ \frac{2AE - BD}{2\sqrt{A}(A - D^2)}\ln w_1 - x - \frac{2\sqrt{A}\,E + B}{2\sqrt{A}\,(\sqrt{A} + D)} + \frac{4ADE - AB - BD^2}{4A(A - D^2)}\ln\left(\frac{\sqrt{A} + D}{2\sqrt{A}}\right) \qquad (115)$$

It is evident from (89), (82), (102), and (103) that

$$\ln\left(\frac{\sqrt{A} + D}{2\sqrt{A}}\right) - \ln w_1 + 2\ln|w_2| = \ln(V + v) \qquad (116)$$

Using (116) and (105) and omitting the two constant terms that follow $-x$ in (115), we obtain

$$I_1 = \left(x + \frac{B - 2DE}{2(A - D^2)}\right)\ln(V + v) + \frac{2AE - BD}{2\sqrt{A}\,(A - D^2)}\ln\left(V + \frac{2Ax + B}{2\sqrt{A}}\right) - x \qquad (117)$$

Expression (117) is I_1 of (59) for the case in which (81) and (82) hold.

VII. THE INTEGRAL OF THE LOGARITHM; $d^2 < 0$

In this section, the integral I_1 of (59) is evaluated for the case in which

$$d^2 < 0 \qquad (118)$$

It is evident from (69) that (118) is accompanied by

$$4AC - B^2 < 0 \qquad (119)$$

Integrating (59) by parts, we obtain

$$I_1 = x\ln w + I_4 \qquad (120)$$

where

$$w = V + v \tag{121}$$

and

$$I_4 = -\int \frac{2DV + 2Ax + B}{2wV}\, x\,dx \tag{122}$$

The integral I_4 will be evaluated by means of the substitution (83). In this substitution, the new variable t is given by

$$t = V + \sqrt{A}\, x \tag{123}$$

Expressions for x and V in terms of w and t are

$$x = \frac{t - w + E}{\sqrt{A} - D} \tag{124}$$

$$V = \frac{\sqrt{A}\, w - Dt - \sqrt{A}\, E}{\sqrt{A} - D} \tag{125}$$

Substitution of (124) and (125) into the numerator of the integrand of I_4 gives

$$I_4 = \frac{1}{2(\sqrt{A} - D)} \int \frac{[2(\sqrt{A} + D)\, t - 2\sqrt{A}\, w + 2\sqrt{A}\, E + B](w - t - E)}{w\, V}\, dx \tag{126}$$

which becomes

$$I_4 = \frac{I_5 - I_6}{2(\sqrt{A} - D)} \tag{127}$$

where

$$I_5 = \int \frac{2(2\sqrt{A} + D)t - 2\sqrt{A}\, w + 4\sqrt{A}\, E + B}{V}\, dx \tag{128}$$

$$I_6 = \int \frac{2(\sqrt{A} + D)t^2 + (4\sqrt{A}\, E + 2DE + B)t + (2\sqrt{A}\, E + B)E}{wV}\, dx \tag{129}$$

In view of (61), substitution of (121) for w and (123) for t in (128) gives

$$I_5 = I_7 + I_8 \tag{130}$$

where

$$I_7 = 2(\sqrt{A} + D) \int dx \tag{131}$$

$$I_8 = \int \frac{4Ax + 2\sqrt{A}\,E + B}{V}\, dx \tag{132}$$

From (131), we obtain

$$I_7 = 2(\sqrt{A} + D)x \tag{133}$$

Replacement of B by 2B − B in the numerator of the integrand of (132) and use of (60) give

$$I_8 = 4I_9 + (2\sqrt{A}\,E - B)\, I_{10} \tag{134}$$

where

$$I_9 = \int \frac{2Ax + B}{2\sqrt{Ax^2 + Bx + C}}\, dx \tag{135}$$

$$I_{10} = \int \frac{dx}{V} \tag{136}$$

Performing the integration in (135), we obtain

$$I_9 = V \tag{137}$$

Substitution of (72) for V in (136) gives

$$I_{10} = \frac{1}{\sqrt{A}} \int \frac{dx}{\sqrt{(x + \frac{B}{2A})^2 + \frac{4AC - B^2}{4A^2}}} \tag{138}$$

Application of [7, Formula 200.01.] to I_{10} gives

$$I_{10} = \frac{1}{\sqrt{A}} \ln \left(x + \frac{B}{2A} + \sqrt{(x + \frac{B}{2A})^2 + \frac{4AC - B^2}{4A^2}}\right) \tag{139}$$

Multiplication of the argument of the logarithm in (139) by a constant

is permissible because it is equivalent to adding a constant to I_{10}. Multiplying the argument of the logarithm by $\sqrt{A}$, we obtain

$$I_{10} = \frac{1}{\sqrt{A}} \ln \left(V + \frac{2Ax + B}{2\sqrt{A}}\right) \tag{140}$$

Substituting (137) and (140) into (134) and then substituting (134) and (133) into (130), we obtain

$$I_5 = 2(\sqrt{A} + D)x + 4V + \frac{(2\sqrt{A}E - B)}{\sqrt{A}} \ln \left(V + \frac{2Ax + B}{2\sqrt{A}}\right) \tag{141}$$

The integration in (129) will be performed with respect to t. After substitution of (86) for dx, (87) for V, and (88) for w, (129) becomes

$$I_6 = 2 \int \frac{2(\sqrt{A} + D)t^2 + (4\sqrt{A}E + 2DE + B)t + (2\sqrt{A}E + B)E}{(\sqrt{A} + D)t^2 + (2\sqrt{A}E + B)t + BE + (\sqrt{A} - D)C} dt \tag{142}$$

Expression (142) reduces to

$$I_6 = 4t + 2I_{11} \tag{143}$$

where

$$I_{11} = \int \frac{(2DE - B)t + (2\sqrt{A}E - B)E - 2(\sqrt{A} - D)C}{(\sqrt{A} + D)t^2 + (2\sqrt{A}E + B)t + BE + (\sqrt{A} - D)C} dt \tag{144}$$

Completing the square in the denominator of (144), we obtain

$$I_{11} = \frac{1}{\sqrt{A} + D} \int \frac{(2DE - B)t + (2\sqrt{A}E - B)E - 2(\sqrt{A} - D)C}{\left(t + \frac{2\sqrt{A}E + B}{2(\sqrt{A} + D)}\right)^2 + \frac{d^2(\sqrt{A} - D)^2}{D^2}} dt \tag{145}$$

where d^2 is given by (90). Expression (145) is rewritten as

$$I_{11} = \frac{2DE - B}{\sqrt{A} + D} I_{12} - \frac{4BDE + 4C(A - D^2) - 4AE^2 - B^2}{2(\sqrt{A} + D)^2} I_{13} \tag{146}$$

where

$$I_{12} = \int \frac{t + \dfrac{2\sqrt{A}\,E + B}{2(\sqrt{A} + D)}}{\left(t + \dfrac{2\sqrt{A}\,E + B}{2(\sqrt{A} + D)}\right)^2 + \dfrac{d^2(\sqrt{A} - D)^2}{D^2}}\, dt \tag{147}$$

$$I_{13} = \int \frac{dt}{\left(t + \dfrac{2\sqrt{A}\,E + B}{2(\sqrt{A} + D)}\right)^2 + \dfrac{d^2(\sqrt{A} - D)^2}{D^2}} \tag{148}$$

From (90), we find that

$$d^2 = \frac{D^2}{4(A - D^2)^2} (4BDE + 4C(A - D^2) - 4AE^2 - B^2) \tag{149}$$

so that (146) reduces to

$$I_{11} = \frac{2DE - B}{\sqrt{A} + D} I_{12} - \frac{2d^2(\sqrt{A} - D)^2}{D^2} I_{13} \tag{150}$$

Recognizing that the numerator of the integrand in (147) is half of the derivative of the denominator, we obtain

$$I_{12} = \frac{1}{2} \ln \left[\left(t + \frac{2\sqrt{A}\,E + B}{2(\sqrt{A} + D)}\right)^2 + \frac{d^2(\sqrt{A} - D)^2}{D^2}\right] \tag{151}$$

Equation (89) reduces (151) to

$$I_{12} = \frac{1}{2} \ln \left[\frac{(V + v)(2\sqrt{A}\,t + B)}{\sqrt{A} + D}\right] \tag{152}$$

Because t is given by (92),

$$2\sqrt{A}\,t + B = 2\sqrt{A}\,V + 2Ax + B \tag{153}$$

Equation (153) reduces (152) to

$$I_{12} = \frac{1}{2} \ln (V + v) + \frac{1}{2} \ln (V + \frac{2Ax + B}{2\sqrt{A}}) + \frac{1}{2} \ln (\frac{2\sqrt{A}}{\sqrt{A} + D}) \tag{154}$$

Omitting the constant term in (154), we obtain

$$I_{12} = \frac{1}{2} \ln (V + v) + \frac{1}{2} \ln (V + \frac{2Ax + B}{2\sqrt{A}}) \tag{155}$$

Application of [7, Formula 120.1.] to (148) gives

$$I_{13} = \frac{D}{d(\sqrt{A} - D)} \tan^{-1} [\frac{D(t + \frac{2\sqrt{A} E + B}{2(\sqrt{A} + D)})}{d(\sqrt{A} - D)}] \tag{156}$$

In (156) and in the remainder of this report, d is the positive square root of the right-hand side of (149), and the value of $\tan^{-1}$ is restricted to lie between $-\pi/2$ and $\pi/2$. Substitution of (92) for t in (156) gives

$$I_{13} = \frac{D}{d(\sqrt{A} - D)} \tan^{-1} w_3 \tag{157}$$

where

$$w_3 = \frac{(2(\sqrt{A} + D)(V + \sqrt{A} x) + 2\sqrt{A} E + B)D}{2d(A - D^2)} \tag{158}$$

Thanks to (155) and (157), (150) becomes

$$I_{11} = \frac{2DE - B}{2(\sqrt{A} + D)} [\ln(V + v) + \ln(V + \frac{2Ax + B}{2\sqrt{A}})] - \frac{2d(\sqrt{A} - D)}{D} \tan^{-1} w_3 \tag{159}$$

Substitution of (159) and (92) into (143) gives

$$I_6 = 4V + 4\sqrt{A} x + \frac{2DE - B}{\sqrt{A} + D} [\ln(V + v) + \ln(V + \frac{2Ax + B}{2\sqrt{A}})]$$

$$- \frac{4d(\sqrt{A} - D)}{D} \tan^{-1} w_3 \tag{160}$$

Subtracting (160) from (141), we obtain

$$I_5 - I_6 = -2(\sqrt{A} - D)x - \frac{2DE - B}{\sqrt{A} + D} \ln(V + v)$$

$$+ \frac{2AE - BD}{\sqrt{A}\,(\sqrt{A} + D)} \ln(V + \frac{2Ax + B}{2\sqrt{A}}) + \frac{4d(\sqrt{A} - D)}{D} \tan^{-1} w_3 \tag{161}$$

Thanks to (161) and (127), (120) becomes

$$I_1 = (x + \frac{B - 2DE}{2(A - D^2)}) \ln(V + v) + \frac{2AE - BD}{2\sqrt{A}\,(A - D^2)} \ln\ (V + \frac{2Ax + B}{2\sqrt{A}})$$

$$- x + \frac{2d}{D} \tan^{-1} w_3 \tag{162}$$

Expression (162) is I_1 of (59) when $d^2 > 0$.

VIII. AN EQUIVALENT EXPRESSION FOR THE INVERSE TANGENT

In this section, an equivalent expression is obtained for the quantity $\tan^{-1} w_3$ that appears in (162). It is evident from (157) that

$$\tan^{-1} w_3 = \frac{d(\sqrt{A} - D)}{D} I_{13} \tag{163}$$

Backtracking, we see that I_{13} is given by (148). Using (86), (87), and (89) to change (148) back into an integral with respect to x, we obtain

$$I_{13} = \frac{\sqrt{A} + D}{2} \int \frac{dx}{V(V + v)} \tag{164}$$

In (164), V and v are given by (60) and (61), respectively. Rationalizing V + v in the denominator of the integrand of (164), we obtain

$$I_{13} = \frac{\sqrt{A} + D}{2} (I_{14} - I_{15}) \tag{165}$$

where

$$I_{14} = \int \frac{dx}{V^2 - v^2} \tag{166}$$

$$I_{15} = \int \frac{v}{(V^2 - v^2)V}\, dx \tag{167}$$

The integral I_{14} is rewritten as

$$I_{14} = \frac{1}{A - D^2} \int \frac{dx}{\left(x + \frac{B - 2DE}{2(A - D^2)}\right)^2 + \frac{d^2}{D^2}} \tag{168}$$

where d^2 is given by (67). Application of [7, Formula 120.1.] to (168) gives

$$I_{14} = \frac{D}{d(A - D^2)} \tan^{-1}\left(\frac{\left(x + \frac{B - 2DE}{2(A - D^2)}\right)D}{d}\right) \tag{169}$$

The integral I_{15} is rewritten as

$$I_{15} = \frac{1}{\sqrt{A}\,(A - D^2)} \int \frac{Dx + F}{\left(\left(x + \frac{B - 2DE}{2(A - D^2)}\right)^2 + \frac{d^2}{D^2}\right)\sqrt{\left(x + \frac{B}{2A}\right)^2 + \frac{4AC - B^2}{4A^2}}}\, dx \tag{170}$$

The case in which

$$\frac{B - 2DE}{A - D^2} = \frac{B}{A} \tag{171}$$

will be considered first. In this case, the substitution

$$T = \sqrt{\left(x + \frac{B}{2A}\right)^2 + \frac{4AC - B^2}{4A^2}} \tag{172}$$

transforms (170) to

$$I_{15} = \frac{D}{\sqrt{A}(A - D^2)} \int \frac{dT}{T^2 + p^2} \tag{173}$$

where

$$p^2 = \frac{d^2}{D^2} - \frac{4AC - B^2}{4A^2} \tag{174}$$

Equations (171) and (69) reduce (174) to

$$p^2 = \frac{d^2}{A} \tag{175}$$

Application of [7, Formula 120.1.] to (173) gives

$$I_{15} = \frac{D}{d(A - D^2)} \tan^{-1} \left(\frac{\sqrt{A}\, T}{d}\right) \tag{176}$$

It is evident from (72) that

$$\sqrt{A}\, T = V \tag{177}$$

so that (176) becomes

$$I_{15} = \frac{D}{d(A - D^2)} \tan^{-1} \left(\frac{V}{d}\right) \tag{178}$$

Expression (178) is I_{15} of (167) for the case in which (171) holds.

Consider I_{15} of (167) for the case in which

$$\frac{B - 2DE}{A - D^2} \neq \frac{B}{A} \tag{179}$$

In this case, I_{15} will be evaluated by means of the substitution [8, Sec. 2.252]

$$x = \frac{C_3 t + C_4}{t + 1} \tag{180}$$

where C_3 and C_4 are constants that will be determined later. Application of this substitution to V of (60) gives

$$V = \sqrt{A\left(\frac{C_3 t + C_4}{t + 1}\right)^2 + B\left(\frac{C_3 t + C_4}{t + 1}\right) + C} \tag{181}$$

which becomes

$$V = \sqrt{\frac{(AC_3^2 + BC_3 + C)t^2 + (2AC_3C_4 + BC_3 + BC_4 + 2C)t + AC_4^2 + BC_4 + C}{(t + 1)^2}} \tag{182}$$

From (60), (61), and (180), we obtain

$$\begin{aligned} V^2 - v^2 = \{ & [(A - D^2)C_3^2 + (B - 2DE)C_3 + C - E^2]t^2 + [2(A - D^2)C_3C_4 \\ & + (B - 2DE)(C_3 + C_4) + 2(C - E^2)]t + (A - D^2)C_4^2 + (B - 2DE)C_4 + C - E^2\}/(t+1)^2 \end{aligned} \tag{183}$$

The constants C_3 and C_4 are determined by annihilating the coefficients of t in (182) and (183). These coefficients will vanish if

$$2AC_3C_4 + B(C_3 + C_4) + 2C = 0 \tag{184}$$

$$D^2C_3C_4 + DE(C_3 + C_4) + E^2 = 0 \tag{185}$$

Equation (185) is rewritten as

$$(C_4D + E)(DC_3 + E) = 0 \tag{186}$$

The above equation will be satisfied if

$$C_3 = -\frac{E}{D} \tag{187}$$

Substitution of (187) for C_3 in (184) gives

$$C_4 = \frac{BE - 2CD}{BD - 2AE} \tag{188}$$

The inequality (179) assures that

$$BD - 2AE \neq 0 \tag{189}$$

With C_3 and C_4 given by (187) and (188), V of (182) becomes

$$V = \frac{1}{|t + 1|} \sqrt{ct^2 + A\left(\frac{BE - 2CD}{BD - 2AE}\right)^2 + B\left(\frac{BE - 2CD}{BD - 2AE}\right) + C} \tag{190}$$

where

$$c = (CD^2 - BDE + AE^2)/D^2 \tag{191}$$

It can be shown that

$$A\left(\frac{BE - 2CD}{BD - 2AE}\right)^2 + B\left(\frac{BE - 2CD}{BD - 2AE}\right) + C = \frac{(4AC - B^2)c}{D^2 b^2} \tag{192}$$

where

$$b = (BD - 2AE)/D^2 \tag{193}$$

Equation (192) reduces (190) to

$$V = \frac{1}{t + 1} \sqrt{ct^2 + \frac{(4AC - B^2)c}{D^2 b^2}} \tag{194}$$

Equation (149) can be recast as

$$d^2 = \frac{D^2}{4(A - D^2)^2} (4(A - D^2)c - b^2 D^2) \tag{195}$$

It is evident from (63), (64), (118) and (195) that

$$c \quad 0 \tag{196}$$

Thanks to (119) and (196), V of (194) can be expressed as

$$V = \frac{\sqrt{c}}{t + 1} \sqrt{t^2 + q^2} \tag{197}$$

where

$$q = \frac{\sqrt{4AC - B^2}}{Db} \tag{198}$$

Substitution of (187) and (188) into (183) gives

$$V^2 - v^2 = \{[(A - D^2)(\frac{E}{D})^2 - (B - 2DE)(\frac{E}{D}) + C-E^2]t^2 + (A-D^2)(\frac{BE - 2CD}{BD - 2AE})^2$$

$$+ (B - 2DE)(\frac{BE - 2CD}{BD - 2AE}) + C - E^2\}/(t + 1)^2 \tag{199}$$

Now,

$$(A - D^2)(\frac{E}{D})^2 - (B - 2DE)(\frac{E}{D}) + C - E^2 = c \tag{200}$$

and

$$(A - D^2)(\frac{BE - 2CD}{BD - 2AE})^2 + (B - 2DE)(\frac{BE - 2CD}{BD - 2AE}) + C-E^2 = \frac{(4AC-B^2)c - 4c^2D^2}{D^2b^2} \tag{201}$$

Hence, (199) reduces to

$$V^2 - v^2 = \frac{c(t^2 + q^2 - p^2)}{(t + 1)^2} \tag{202}$$

where

$$p = \frac{2\sqrt{c}}{b} \tag{203}$$

and q is given by (198). Equation (149) can be recast as

$$d^2 = \frac{D^2}{4(A - D^2)^2} (4AC - B^2 - 4D^2c) \tag{204}$$

It is evident from (198), (203), (204) and (118) that

$$q^2 \quad p^2 \tag{205}$$

From (180), we obtain

$$dx = \frac{C_3 - C_4}{(t + 1)^2} dt \tag{206}$$

Substitution of (187) and (188) into (206) gives

$$dx = \frac{2c}{Db(t + 1)^2} dt \tag{207}$$

where b and c are given by (193) and (191), respectively. Substituting

(180) for x in (61) and using (187), (188), (191), and (193), we obtain

$$v = \frac{-2c}{b(t+1)} \tag{208}$$

Substitution of (197) for V, (202) for $V^2 - v^2$, (207) for dx, and (208) for v in (167) yields

$$I_{15} = -\frac{4\pi\bar{c}}{Db^2}\int \frac{t+1}{|t+1|(t^2+q^2-p^2)\sqrt{t^2+q^2}}\,dt \tag{209}$$

Assuming that t + 1 does not change sign during the integration in (209), we obtain

$$I_{15} = -F \qquad , \qquad t > -1 \tag{210}$$

$$I_{15} = F \qquad , \qquad t < -1 \tag{211}$$

where

$$F = -\frac{4\pi\bar{c}}{Db^2}\int \frac{dt}{(t^2+q^2-p^2)\sqrt{t^2+q^2}} \tag{212}$$

Avoiding the question as to whether or not the combination of (210) and (211) is a valid representation for I_{15} of (209), we proceed to evaluate (212) and transform (210) and (211) into the domain of x.

The substitution

$$T = \frac{t}{\sqrt{t^2+q^2}} \tag{213}$$

and subsequent equations

$$dT = \frac{q^2}{(t^2+q^2)^{3/2}}\,dt \tag{214}$$

$$t^2 + q^2 = \frac{q^2}{1-T^2} \tag{215}$$

reduce (212) to

$$F = -\frac{4\sqrt{c}}{Db^2p^2}\int\frac{dT}{T^2+\frac{q^2-p^2}{p^2}} \tag{216}$$

Application of [7, Formula 120.1.] to (216) gives

$$F = -\frac{4\sqrt{c}}{Db^2p\sqrt{q^2-p^2}}\tan^{-1}\left(\frac{pT}{\sqrt{q^2-p^2}}\right) \tag{217}$$

In (217), T is given by (213).

With a view toward expressing T of (213) in terms of x, we use (208) to obtain

$$t + 1 = -\frac{2c}{bv} \tag{218}$$

It is evident from (218) that

$$t = -\frac{bv+2c}{bv} \tag{219}$$

Equation (197) gives

$$\sqrt{t^2+q^2} = \frac{|t+1|V}{\sqrt{c}} \tag{220}$$

Substituting (218) for t+1 in (220), we obtain

$$\sqrt{t^2+q^2} = \frac{2\sqrt{c}\,V}{|bv|} \tag{221}$$

Thanks to (219) and (221), (213) becomes

$$T = -\frac{bv\,(bv+2c)}{2|bv|\sqrt{c}\,V} \tag{222}$$

Before substituting (222) into (217), we use (198), (203), and (204) to obtain

$$\sqrt{q^2-p^2} = \frac{2(A-D^2)d}{D^2|b|} \tag{223}$$

Now, (222), (223), and (203) transform (217) to

$$F = -\frac{bD}{|b|(A - D^2)d}\tan^{-1}\left(\frac{-v(bv + 2c)D^2}{2|v|dV(A - D^2)}\right) \tag{224}$$

which is equivalent to

$$F = \frac{bvD}{|bv|(A - D^2)d}\tan^{-1}G \tag{225}$$

where

$$G = \frac{(bv + 2c)D^2}{2dV(A - D^2)} \tag{226}$$

In view of (196), it is evident from (218) that $bv \;\; 0$ when $t \;\; -1$ and that $bv \;\; 0$ when $t \;\; -1$. Hence, with F given by (225), (210) becomes

$$I_{15} = -\frac{D}{(A - D^2)d}\tan^{-1}G, \quad bv \;\; 0 \tag{227}$$

and (211) becomes

$$I_{15} = -\frac{D}{(A - D^2)d}\tan^{-1}G, \quad bv \;\; 0 \tag{228}$$

The combination of (227) and (228) will be a valid representation for I_{15} of (167) if this combination is continuous at bv = 0. Since the right-hand sides of (227) and (228) are identical, the combination of (227) and (228) is obviously continuous at bv = 0. Hence, this combination is a valid representation for I_{15} of (167). This representation is

$$I_{15} = -\frac{D}{(A - D^2)d}\tan^{-1}G \tag{229}$$

If

$$g = \tan^{-1} G \tag{230}$$

then

$$G = \tan g \tag{231}$$

Now, (231) implies that

$$\tan(\frac{\pi}{2} - g) = \frac{1}{G} \tag{232}$$

Because g lies between $-\pi/2$ and $\pi/2$, $\frac{\pi}{2} - g$ must lie between 0 and π. It is now evident from (232) that

$$\frac{\pi}{2} - g = \tan^{-1*}(\frac{1}{G}) \tag{233}$$

where $\tan^{-1*}$ is the inverse tangent whose value lies between 0 and π. From (233), we obtain

$$g = \frac{\pi}{2} - \tan^{-1*}(\frac{1}{G}) \tag{234}$$

Substitution of (230) for g in (234) gives

$$\tan^{-1} G = \frac{\pi}{2} - \tan^{-1*}(\frac{1}{G}) \tag{235}$$

Substituting (235) for $\tan^{-1} G$ in (229), omitting the constant $\pi/2$, and using (226), we obtain

$$I_{15} = \frac{D}{(A - D^2)d} \tan^{-1*}(\frac{2dV(A - D^2)}{(bv + 2c)D^2}) \tag{236}$$

Expression (236) is I_{15} of (167) for the case in which (179) holds. If (236) were to reduce to (178) when (171) holds, then (236) would also be I_{15} of (167) for the case in which (171) holds. The following reasoning is presented to show that (236) reduces to (178) when (171)

is true. It is evident from (171) and (193) that

$$b = 0 \tag{237}$$

so that (195) reduces to

$$d^2 = \frac{D^2 c}{A - D^2} \tag{238}$$

Solving (238) for c, substituting this value of c into (236), and using (237), we obtain (178). Therefore, (236) reduces to (178) when (171) holds. Hence, expression (236) is I_{15} of (167) for both the case in which (171) holds and the case in which (179) holds.

Concerning the argument of $\tan^{-1}$ in (169), we find that

$$\left(x + \frac{B - 2DE}{2(A - D^2)}\right) D = u \tag{239}$$

where

$$u = v + \frac{bD^2}{2(A - D^2)} \tag{240}$$

where v and b are given by (61) and (193), respectively. Hence, (169) becomes

$$I_{14} = \frac{D}{d(A - D^2)} \tan^{-1}\left(\frac{u}{d}\right) \tag{241}$$

Substituting (236) and (241) into (165) and then substituting (165) into (163), we find that, within an additive constant,

$$\tan^{-1} w_3 = \frac{1}{2} \left[\tan^{-1}\left(\frac{u}{d}\right) - \tan^{-1*} \left(\frac{2dV(A - D^2)}{(bv + 2c)D^2}\right)\right] \tag{242}$$

IX. A GLOBAL FORM FOR THE INTEGRAL OF THE LOGARITHM

Equations (193), (239), and (242) allow (162) to be rewritten as

$$I_1 = \frac{1}{D} [u \ln(V + v) - \frac{bD^3}{2\sqrt{A}\,(A - D^2)} \ln (V + \frac{2Ax + B}{2\sqrt{A}}) - Dx$$

$$+ d \tan^{-1}(\frac{u}{d}) - d\tan^{-1*}(\frac{2dV(A - D^2)}{(bv + 2c)D^2})] \qquad (243)$$

Expression (243) is I_1 of (59) when $d^2 > 0$. In this section, it is shown that (243) is a global form for the integral I_1 of (59).

Expression (243) will be a global form for the integral I_1 of (59) if (243) is I_1 of (59) in all possible cases. Only three cases are possible, the case in which (118) holds, the case in which (81) and (82) hold, and the case in which (68) holds. By definition, (243) is I_1 of (59) for the case in which (118) holds. It remains to be shown that (243) is I_1 of (59) for the case in which (81) and (82) hold and the case in which (68) holds.

Expression (243) will be I_1 of (59) for the case in which (81) and (82) hold if (243) reduces to (117) when (81) and (82) hold. Setting $d = 0$ in (243) and noting that $\tan^{-1}(u/d)$ can not exceed $\pi/2$, we obtain

$$I_1 = \frac{1}{D} [u \ln(V + v) - \frac{bD^3}{2\sqrt{A}(A - D^2)} \ln (V + \frac{2Ax + B}{2\sqrt{A}}) - Dx] \qquad (244)$$

In view of (193) and (239), it is apparent that (244) is equal to (117). Therefore, (243) is I_1 of (59) for the case in which (81) and (82) hold.

Consider the argument $V + v$ of the first logarithm in (244). It is evident from (72) and (81) that $V > 0$. Hence, if $v \geq 0$, then $V + v > 0$. If $v < 0$, we write

$$V + v = \frac{V^2 - v^2}{V - v} \tag{245}$$

In view of (82), comparison of (168) with (166) gives

$$V^2 - v^2 = (A - D^2)(x + \frac{B - 2DE}{2(A - D^2)})^2 \tag{246}$$

Equation (239) reduces (246) to

$$V^2 - v^2 = \frac{(A - D^2)u^2}{D^2} \tag{247}$$

Substitution of (247) into (245) gives

$$V + v = \frac{(A - D^2)u^2}{D^2(V - v)} \tag{248}$$

According to (248), if $V + v$ were to approach zero, then u would approach zero simultaneously so that the $u \ln (V + v)$ term in (244) would approach zero harmlessly. However, if u is exactly zero, we must agree to omit the $u \ln(V + v)$ term from (244) without calculating $\ln(V + v)$.

Expression (243) will be I_1 of (59) for the case in which (68) holds if (244) reduces to (80) when (68) holds. Equation (68) is accompanied by (70), so that, according to (193),

$$b = 0 \tag{249}$$

Thanks to (249), the coefficient of the second logarithm term in (244) is zero. Unfortunately, the argument of this logarithm could be zero. If, because its coefficient is zero, we agree to set this logarithm term equal to zero without calculating the logarithm, then (244) reduces to

$$I_1 = \frac{1}{D}(u \ln(V + v) - Dx) \tag{250}$$

Equation ([illegible]) reduces [illegible]

$$\frac{(2Ax + B)D}{2A} = u \tag{251}$$

Substitution of (251) into (250) yields

$$I_1 = \frac{2Ax + B}{2A} \ln(V + v) - x \tag{252}$$

which is identical to (80). Hence, (243) is I_1 of (59) for the case in which (68) holds. It is evident from (78) that, whenever $V + v$ approaches zero, $2Ax + B$ also approaches zero so that the logarithm term in (252) approaches zero harmlessly. Nevertheless, if $2Ax + B$ is exactly zero, we must agree to omit the logarithm term from (252) without calculating $\ln(V + v)$.

It has been established that (243) is I_1 of (59) in all three possible cases, the case in which (118) holds, the case in which (81) and (82) hold, and the case in which (68) holds. Therefore, (243) is a global form for the integral I_1 of (59).

X. USE OF THIS INTEGRAL TO EVALUATE THE POTENTIAL

In this section, the global form (243) for the integral I_1 of (59) is used to evaluate the potential $V(r)$ of (58).

Replacing the constants A,B,C,D, and E inherent in (59) and (243) by A_i, B_i, C_i, D_i, and E_i, we obtain

$$\int_0^1 (\ln (R_i + x_i))d\zeta = \frac{1}{D_i} [u_i \ln(R_i + x_i)$$

$$- \frac{b_i D_i}{2\sqrt{A_i}\,(a_i - 1)} \ln(R_i + \frac{2A_i\zeta + B_i}{2\sqrt{A_i}}) - D_i\zeta$$

$$+ d_i \tan^{-1}(\frac{u_i}{d_i}) - d_i \tan^{-1*}(\frac{2d_i R_i (a_i - 1)}{b_i x_i + 2c_i})]_0^1 \tag{253}$$

where R_i and x_i are given by (32) and (36), respectively. Also, d_i is the positive square root of the right-hand side of (54). Furthermore,

$$a_i = A_i/D_i^2 \tag{254}$$

$$b_i = (B_i D_i - 2A_i E_i)/D_i^2 \tag{255}$$

$$c_i = (C_i D_i^2 - B_i D_i E_i + A_i E_i^2)/D_i^2 \tag{256}$$

$$u_i = x_i + \frac{b_i}{2(a_i - 1)} \tag{257}$$

Equations (255) - (257) are similar to (193), (191), and (240), respectively. It is evident from (195) that an alternative expression for d_i^2 is

$$d_i^2 = \frac{c_i}{a_i - 1} - \frac{b_i^2}{4(a_i - 1)^2} \tag{258}$$

In (253), $\tan^{-1}$ is the inverse tangent whose value is restricted to lie between $-\pi/2$ and $\pi/2$, and $\tan^{-1*}$ is the inverse tangent whose value is restricted to lie between 0 and π. If $d_i = 0$, then the $\tan^{-1}$ term in (253) is to be set equal to zero. If the coefficient of one of the logarithm terms in (253) is zero, then that logarithm term is to be set equal to zero.

When (253) is substituted into (58), the term in (253) cancels out of the sum on i and we are left with

$$V(\underline{r}) = \frac{|(\underline{r}_2 - \underline{r}_1) \times (\underline{r}_3 - \underline{r}_1)|}{4\pi\varepsilon_o |\underline{r}_2 - \underline{r}_1|} \sum_{i=1}^{2} \frac{(-1)^i}{D_i} [u_i \ln(R_i + x_i) + L_i$$

$$+ d_i \tan^{-1}\left(\frac{u_i}{d_i}\right) - d_i \tan^{-1*}\left(\frac{2d_i R_i (a_i - 1)}{b_i x_i + 2c_i}\right)]_0^1 \tag{259}$$

where

$$L_i = -\frac{b_i D_i}{2\sqrt{A_i}\,(a_i - 1)} \ln \left(R_i + \frac{2A_i\zeta + B_i}{2\sqrt{A_i}}\right) \tag{260}$$

XI. MANIPULATION OF THE POTENTIAL INTO THE DESIRED FORM

In this section, the term L_i in expression (259) for the potential $V(\underline{r})$ is manipulated into the second logarithm term of [5, Eq. (4)]. Then, (259) is rewritten so as to coincide with [5, Eq. (4)].

Expression (260) for L_i is recast as

$$L_i = -\frac{b_i D_i}{2\sqrt{A_i}\,(a_i - 1)} \ln \left(\frac{2\sqrt{A_i}\,R_i}{D_i} + \frac{2A_i\zeta + B_i}{D_i}\right) + K_{1i} \tag{261}$$

where

$$K_{1i} = \frac{b_i D_i}{2\sqrt{A_i}\,(a_i - 1)} \ln \left(\frac{2\sqrt{A_i}}{D_i}\right) \tag{262}$$

It can be shown that

$$\frac{2A_i\zeta + B_i}{D_i} = 2a_i x_i + b_i \tag{263}$$

where x_i, a_i, and b_i are given by (36), (254), and (255), respectively. Substitution of (263) into (261) gives

$$L_i = -\frac{b_i D_i}{2\sqrt{A_i}\,(a_i - 1)} \ln\left(\frac{D_i}{|D_i|}\left(\frac{2\sqrt{A_i}\,R_i}{D_i} + 2a_i x_i + b_i\right)\right) + K_{1i} \tag{264}$$

For the purpose of obtaining still another form of L_i of (260), we call the argument of the logarithm in (260) ARG and express it as

$$\mathrm{ARG} = \frac{\left(R_i + \dfrac{2A_i\zeta + B_i}{2\sqrt{A_i}}\right)\left(R_i - \dfrac{2A_i\zeta + B_i}{2\sqrt{A_i}}\right)}{R_i - \dfrac{2A_i\zeta + B_i}{2\sqrt{A_i}}} \tag{265}$$

With R_i given by (32), (265) reduces to

$$ARG = \frac{4A_iC_i - B_i^2}{4A_i\left(R_i - \frac{2A_ix_i + B_i}{2\sqrt{A_i}}\right)} \tag{266}$$

If $4A_iC_i - B_i^2 = 0$, then it can be inferred from (70) and (193) that $b_i = 0$ in which case L_i is to be set equal to zero. Hence, it is permissible to assume that $4A_iC_i - B_i^2 \neq 0$. In this case, (45) implies that

$$4A_iC_i - B_i^2 > 0 \tag{267}$$

Equation (267) assures that both the numerator and the denominator in (266) are positive. Now, (266) allows (260) to be expressed as

$$L_i = \frac{b_iD_i}{2\sqrt{A_i}\,(a_i - 1)} \ln\left(\frac{2\sqrt{A_i}\,R_i}{D_i} - \frac{2A_ix_i + B_i}{D_i}\right) + K_{2i} \tag{268}$$

where

$$K_{2i} = -\frac{b_iD_i}{2\sqrt{A_i}\,(a_i - 1)} \ln\left(\frac{4A_iC_i - B_i^2}{2\sqrt{A_i}\,D_i}\right) \tag{269}$$

Substitution of (263) into (268) gives

$$L_i = \frac{b_iD_i}{2\sqrt{A_i}\,(a_i - 1)} \ln\left(\frac{2\sqrt{A_i}\,R_i}{D_i} - \frac{D_i(2a_ix_i + b_i)}{D_i}\right) + K_{2i} \tag{270}$$

It is permissible to omit the constants K_{1i} and K_{2i} in (264) and (270) because they will cancel out when the limits of integration are applied. Taking (264) for $D_i < 0$, taking (270) for $D_i > 0$, and omitting the constants K_{1i} and K_{2i}, we obtain

$$L_i = - \frac{b_i}{2\sqrt{a_i}\,(a_i - 1)} \ln\,(2\sqrt{a_i}\,R_i + 2a_i x_i + b_i) \tag{271}$$

In (271), $\sqrt{a_i}$ is, of course, the positive square root of a_i.

Substituting (271) into (259), we conclude that

$$\begin{aligned} V(\underline{r}) = \frac{(\underline{r}_2 - \underline{r}_1) \cdot (\underline{r}_3 - \underline{r}_1)}{4\pi\epsilon_o\,|\underline{r}_2 - \underline{r}_1|} \sum_{i=1}^{2} \frac{(-1)^i}{D_i} \Big[& u_i \ln(R_i + x_i) \\ & - \frac{b_i}{2\sqrt{a_i}\,(a_i - 1)} \ln\,(2\sqrt{a_i}\,R_i + 2a_i x_i + b_i) \\ & + d_i \tan^{-1}\left(\frac{u_i}{d_i}\right) - d_i \tan^{-1*}\left(\frac{2d_i R_i (a_i - 1)}{b_i x_i + 2c_i}\right)\Big]_0^1 \end{aligned} \tag{272}$$

where R_i, x_i, a_i, b_i, c_i, and u_i are given by (32), (36), and (254)-(257), respectively. Moreover, d_i is the positive square root of the right-hand side of (258). In (272), $\tan^{-1}$ is the inverse tangent whose value is restricted to lie between $-\pi/2$ and $\pi/2$, and $\tan^{-1*}$ is the inverse tangent whose value is restricted to lie between 0 and π. If $d_i = 0$, then the $\tan^{-1}$ term in (272) is to be set equal to zero. If the coefficient of one of the logarithms in (272) is zero, then that logarithm term is to be set equal to zero. Except for the facts that there are no magnitude signs on the argument of its second logarithm and that its last term contains an asterisk, the right-hand side of (272) is identical to the right-hand side of [5, Eq. (4)]. There is no need to take the magnitude of the argument of the second logarithm in (272) because this argument can never be negative. The second inverse tangent in [5, Eq. (4)] has to be on the range from 0 to π in order to pass continuously through $\pi/2$ as $b_i x_i + 2c_i$ passes through zero.

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